An Infinite Sequence
of Approximate Angle Trisections

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We begin with the simple-minded approximate trisection described in Underwood Dudley’s “The Trisectors” on pp. 116–118:

The angle to trisect is \( \alpha = \angle AOB \), where \(|OA| = |OB|\), and points C and D trisect the line segment AB. The angle \( \tau(\alpha) = \angle AOC = \arctan\left(\frac{\sin\alpha}{2 + \cos\alpha}\right) \) approximates one third of the angle \( \alpha \). The approximation is not very good, as is apparent from the diagram of \( \text{err}(\alpha) = \alpha/3 - \tau(\alpha) \):

\[
\text{err}(\alpha) = \frac{\alpha^3}{81} + \frac{\alpha^5}{972} + \frac{7\alpha^7}{87480} + \frac{443\alpha^9}{79361856} + \frac{1291\alpha^{11}}{3968092800} + O(\alpha^{13}).
\]

Both \( \tau(\alpha) \) and \( \text{err}(\alpha) \) are analytic functions of \( \alpha \) on the interval \((-\pi, \pi)\). The power series expansion of \( \text{err}(\alpha) \) at 0 begins as follows:

1For any points \( P \neq Q \neq R \) of the oriented Euclidean plane, we denote by \( \angle PQR \) the measured angle \( \varphi \) (measured in radians) from the vector \( \overrightarrow{QP} \) to the vector \( \overrightarrow{QR} \), lying in the range \(-\pi < \varphi \leq \pi\).
\[\delta = \frac{3^\circ}{2^m},\] where \(m\) is a large positive integer, we can get excellent approximations. We will use the reduction trick with other approximate trisections, but we shall avoid ridiculously small angles \(\delta\) and will instead stick to \(\delta = 15^\circ\), always reducing angles modulo \(3\delta = 45^\circ\).

Now we extend our first approximate trisection construction, adding an arc and some lines and points based on it. We draw the arc AB with center O, extend the line segments OC and OD to the points E resp. G on the arc, and construct the point F on the arc EG so that \(|GF| = |GB|\) (and hence also \(|GF| = |AE|\)). The point \(\Omega\) on the arc AB exactly trisects this arc, that is, \(\angle AO\Omega = \alpha/3\); this point is of course not constructible by straightedge and compass (in general); we put it there to mark the place of the unattainable ideal, and to draw attention to the fact that \(\Omega\) trisects also the arc EF, since \(\angle EO\Omega = \frac{1}{2}\angle EOF\).

Construction of the point \(F\) can be simplified by noting that the line segment CF is parallel (in the same direction) to the line segment OD: we just draw the parallel to OD through C, producing it to the point F on the arc AB. To help us prove CF parallel to OD, we first draw the line segment FD. Since \(\angle FDG = \angle GDB = \angle ODC = \angle DCO\), we have \(\angle CDF = \angle COD\); since \(|FD| = |CD|\), the triangle \(\triangle CFD\) is similar to the triangle \(\triangle CDO\), whence \(\angle FCD = \angle DCO = \angle GDB\), which completes the proof. As a bonus we have obtained the identity \(|OC| \cdot |CF| = |CD|^2\). An obvious (but useful) consequence is that \(\tau(\alpha) < \frac{1}{3}\alpha\) for \(0 < \alpha \leq \pi\).

In the second approximate trisection we just repeat the first approximate trisection on the gap \(\angle EOF\) left after the first approximate trisection has been applied to \(\angle AOB\). This makes perfect sense, because the exact trisection point \(\Omega\) of the arc AB exactly trisects the arc EF.

The points C and D trisect the line segment AB, the line CF is parallel to the line OD, and the point T is on the line segment EF, \(|ET| = |EF|/3\); the line segment EF is actually drawn in the figure, though it is almost indistinguishable from the arc EF. The angle \(\tau_2(\alpha) = \angle AOT\)
is our second approximation of one third of the angle \( \alpha = \angle AOB \). (The subscript 2 indicates the second approximation; the first approximation is \( \tau_1(\alpha) = \tau(\alpha) \).) The error of the second construction, \( err_2(\alpha) = \alpha/3 - \tau_2(\alpha) \), is considerably smaller than the error \( err_1(\alpha) = err(\alpha) \):

![Graph showing \( err(\alpha) \) vs. \( \alpha \) in degrees]

Both \( \tau_2(\alpha) \) and \( err_2(\alpha) \) are analytic functions of \( \alpha \) on the interval \((-\pi..\pi)\), and the power series expansion of \( err_2(\alpha) \) at 0 begins as follows:

\[
err_2(\alpha) = \frac{\alpha^9}{3^{13}} + \frac{\alpha^{11}}{4 \cdot 3^{13}} + \frac{29\alpha^{13}}{80 \cdot 3^{15}} + \frac{6479\alpha^{15}}{560 \cdot 3^{29}} + \frac{90211\alpha^{17}}{22400 \cdot 3^{31}} + O(\alpha^{19}).
\]

The absolute value \( |err_2(\alpha)| \) of the error for \(|\alpha| \leq 22.5^\circ\) is less than 0.000 03″; we can maintain this precision for any angle by reducing it modulo 45° to the remainder in the interval \([-22.5^\circ..22.5^\circ)\), and approximately trisecting the remainder; the error on a circle with the radius that of Earth is then less than 1 mm.

Applying, once again, the first approximate trisection to the gap left after the second approximate trisection, we obtain the third approximate trisection. In the following figure we renamed the points appearing in the construction to highlight the phases of the construction: in the \( k \)-th phase we trisect the line segment \( A_kB_k \) at the points \( C_k \) and \( D_k \).

![Diagram showing the construction phases]

We are forced to observe the third phase of the construction under a microscope if we want to tell apart the constructed points. The right panel of the figure shows the construction of the points \( A_3, B_3, \) and \( C_3 \), magnified 900-fold: the downward-slanted lines on lower left and upper right are tiny fragments of the segment \( A_2B_2 \) resp. the arc \( A_2B_2 \); the segment \( C_2A_3 \) is a continuation of the segment \( OC_2 \), the segment \( C_2B_3 \) is parallel to the segment \( OD_2 \), and the point \( C_3 \) is at one third of the line segment \( A_3B_3 \). The approximations of one third of the given angle \( \alpha = \angle AOB \) after successive construction’s phases are \( \tau_1(\alpha) = \angle AOC_1 \), \( \tau_2(\alpha) = \angle AOC_2 \), and \( \tau_3(\alpha) = \angle AOC_3 \).

The power series expansion of \( err_3(\alpha) = \alpha/3 - \tau_3(\alpha) \) at 0 begins

\[
err_3(\alpha) = \frac{\alpha^{27}}{3^{40}} + \frac{\alpha^{29}}{4 \cdot 3^{38}} + \frac{37\alpha^{31}}{40 \cdot 3^{41}} + \frac{150089\alpha^{33}}{2240 \cdot 3^{46}} + \frac{543671\alpha^{35}}{11200 \cdot 3^{49}} + O(\alpha^{37}).
\]
The maximum error $|err_3(\alpha)|$ for $|\alpha| \leq 22.5^\circ$ is less than $(2.1 \cdot 10^{-25})''$; on a circle with the radius that of the Milky Way galaxy (50,000 light years), this maximum error amounts to less than 5Å, which is less than five diameters of a hydrogen atom.

If we repeat the first approximate trisection on the gap left after the third approximate trisection, then the power series expansion of the resulting error $err_4(\alpha)$ begins

$$err_4(\alpha) = \frac{\alpha^{84}}{3^{121}} + \frac{\alpha^{83}}{4 \cdot 3^{119}} + \frac{209 \alpha^{85}}{80 \cdot 3^{121}} + \frac{141871 \alpha^{87}}{280 \cdot 3^{126}} + \frac{1049759 \alpha^{89}}{1120 \cdot 3^{127}} + O(\alpha^{91}),$$

and the maximum error $|err_4(\alpha)|$ for $|\alpha| \leq 22.5^\circ$ is less than $\left(7.131 \cdot 10^{-86}\right)''$. While we still managed to somehow visualise maximum errors of the first three approximate trisections (with the compulsory reduction modulo $45^\circ$), the smallness of the maximum error of the fourth approximate trisection is already beyond our comprehension. Nevertheless, try to imagine a circle whose radius is the radius of the observable universe (currently estimated at about 46 billion light years); then the error on the circle this large is only $1.5 \cdot 10^{-64}$ m, which is less than $10^{42}$-th part of the (quantum-theoretic) electron’s diameter ($2 \cdot 10^{-22}$ m); or, put differently, if the true radius of our universe were a million billion billion billion billion times larger than the presumed radius of that tiny tiny tiny dot in it which we can observe (or so we believe), then the maximum error on a circle with the radius equal to the radius of this huge universe would be less than the diameter of an electron.

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And so it goes on: we keep applying the first, basic approximate trisection to the gap left after each subsequent trisection to obtain the next, (much much) more precise approximate trisection. The power series expansion of the error $err_n(\alpha)$ of the $n$-th approximate trisection starts with the term $\alpha^{3^9}/3^{(3^{n+1}-1)/2}$, followed by $b_n\alpha^{3^{n+2}} + c_n\alpha^{3^n+4} + \ldots$.

We are entering the second part of this essay. In the first part we actually looked at the first three approximate trisections, and commented on the magnitude of error of the fourth one. From here on we shall study the behavior of the sequence of errors $err_n(\alpha)$, $n = 1, 2, 3, \ldots$

We shall denote by $\gamma_n(\alpha)$ the gap left after the $n$-th approximate trisection. With the basic approximate trisection we have the following formulas for the approximate third $\tau(\alpha)$, the gap $\gamma(\alpha) = \gamma_1(\alpha)$, and the error $err(\alpha)$:

$$\tau(\alpha) = \arctan \frac{\sin \alpha}{2 + \cos \alpha}, \quad \gamma(\alpha) = \alpha - 3\tau(\alpha), \quad err(\alpha) = \frac{\gamma(\alpha)}{3}.$$  

The gap, the error, and the approximate third for the $n$-th approximate trisection are then

$$\gamma_n(\alpha) = \gamma^n(\alpha), \quad err_n(\alpha) = \frac{\gamma_n(\alpha)}{3}, \quad \tau_n(\alpha) = \frac{\alpha}{3} - err_n(\alpha),$$

where $\gamma^n$ denotes the $n$-th iterate (composition power) of the function $\gamma$, defined for any integer $n \geq 0$: $\gamma^0(\alpha) = \alpha$, $\gamma^1(\alpha) = \gamma(\alpha)$, $\gamma^2(\alpha) = \gamma(\gamma(\alpha))$, $\ldots$, $\gamma^{n+1}(\alpha) = \gamma(\gamma^n(\alpha))$.  

The function $\tau(\alpha)$ is defined for all real $\alpha$; it is an analytic function of a real variable, and it is odd and periodic with the period $2\pi$. The power series expansion of $\tau(\alpha)$ at $0$ begins

$$\tau(\alpha) = \frac{\alpha}{3} - \frac{\alpha^3}{81} - \frac{\alpha^5}{972} - \frac{7\alpha^7}{87480} + O(\alpha^9).$$

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2Where $b_n = 1/(4 \cdot 3^{3^{n+1}+3}/2^{n-2})$ and $c_n = (5 \cdot 3^n + 13)/(160 \cdot 3^{3^{n+1}+7}/2^{n-2})$.

3We shall write the $n$-th powers of, say, $\sin \theta$ or $\ln x$, as $(\sin \theta)^n$ and $(\ln x)^n$, never as $\sin^n \theta$ or $\ln^n x$, since the latter will always denote iterates: $\sin^3 \theta = \sin \sin \sin \theta$, $\ln^2 x = \ln \ln x$.  

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What is the convergence radius of this power series? The easiest way to answer this question is to observe \( \tau(z) \) as a function of complex variable and to locate its singularity in the complex plane that is nearest to the origin. The \( \arctan(z) \) function has logarithmic singularities at \( z = \pm i \). Solving \( (\sin z)/(2 + \cos z) = \pm i \) for \( z \) gives us the locations \( (2k+1)\pi \pm i \ln 2, k \in \mathbb{Z}, \) of (logarithmic) singularities of the function \( \tau(z) \). The convergence radius of the power series is therefore \( \sqrt{\pi^2 + (\ln 2)^2} \). The diagram of the real part of three branches of \( \tau(z) \) clearly shows the logarithmic character of the singularities at \( -\pi + i \ln 2 \) and \( \pi + i \ln 2 \):

The imaginary part of \( \tau(z) \) is a single value, which decreases to \(-\infty\) as \( z \) approaches a singularity.

Next, we examine the gap \( \gamma(z) \) as a function of a complex variable \( z \). Let us start by

\[\text{Here they meet again, as if by accident, the mathematical constants } \pi \text{ and } \ln 2. \text{ They will have to be more careful, or people will start talking.}\]
determining all zeros of $\gamma(z)$.\textsuperscript{5} We have $\gamma(z) = 3k\pi$ for some $k \in \mathbb{Z}$ if and only if

$$\tan \frac{z}{3} = \frac{\sin z}{2 + \cos z}. \tag{1}$$

Set $w = z/3$. Substituting $\sin z = (4(\cos w)^2 - 1)\sin w$ and $\cos z = 4(\cos w)^3 - 3\cos w$ into the equation (1), we get the equation

$$\frac{\sin w}{\cos w} = \frac{(4(\cos w)^2 - 1)\sin w}{2 + 4(\cos w)^3 - 3\cos w} \tag{2}$$

When the denominator $\cos w$ on the left hand side is 0, the denominator on the right hand side is 2, thus the two denominators are never both zero, and hence the two sides of (2) are never both infinite. After simple manipulation the equation (2) simplifies to the equivalent equation

$$(1 - \cos w)\sin w = 0,$$

which has the solutions $w = k\pi$, $k \in \mathbb{Z}$, whence $z = 3k\pi$, $k \in \mathbb{Z}$, are all the zeros of $\gamma(z)$.

The branches of the multi-valued function $\gamma(z)$ are glued together along the branch cuts \{(2$k+1)\pi \pm i(t + \ln 2) \mid t \geq 0\}, k \in \mathbb{Z}$; the principal branch is the one on which $\gamma(0) = 0$. Let $S$ be the open strip $\{z \in \mathbb{C} \mid -\pi < \Re z < \pi \}$. From now on we shall observe only the restriction of the principal branch of $\gamma$ to $S$, which we shall continue to denote $\gamma$. So now we have $\gamma \in H(S)$, where $H(S)$ denotes the set of all functions holomorphic on $S$.

Let $R(3.1, 5.1)$ be the rectangle in the complex plane consisting of all $z$ with $|\Re z| \leq 3.1$ and $|\Im z| \leq 5.1$. Here we see how $\gamma$ maps the first quadrant of this rectangle:

Since $\gamma(z)$ is real for $z$ real, it preserves complex conjugation, that is, $\gamma(\overline{z}) = \overline{\gamma(z)}$; since $\gamma(z)$ is an odd function, it also preserves mirroring across the imaginary axis, that is, $\gamma(-\overline{z}) = -\gamma(\overline{z})$; we conclude that $\gamma$ maps the rectangle $R(3.1, 5.1)$ into itself. Now we are going to trust what we can see with our mind’s eyes and state the following fact: for every $a$, $0 < a < \pi$, there exists $A = A(a) > 0$, so that $\gamma$ maps the rectangle $R(a, A)$, consisting of all $z$ with $|\Re z| \leq a$ and $|\Im z| \leq A$, into itself; moreover, $A(a) \to +\infty$ as $a \to \pi$. (It would take us a day or so of drudgery to prove this fact, but we will not bother to actually do it. Anyway, we know

\textsuperscript{5}The function $\gamma(z)$ is multi-valued: if $w$ is one of its values at $z$, then $w + 3k\pi$, $k \in \mathbb{Z}$, are all its values at $z$. We consider $z_0$ a zero of $\gamma(z)$ if 0 is one of its values at $z_0$.}
that what we claim is true.) It follows that for any \( n \in \mathbb{N} \) the iterate \( \gamma^n \) maps the rectangle \( R(a, A(a)) \) into itself.

Another simple, but important, consequence is that \( \gamma \) maps the strip \( S \) into itself, and therefore every iterate \( \gamma^n, n \in \mathbb{N} \), is defined and holomorphic on \( S \), from which it follows that the functions \( \text{err}_n(z) = \gamma^n(z)/3 \) and \( \tau_n(z) = z/3 - \text{err}_n(z) \) (with the restricted function \( \gamma \)) are defined and holomorphic on \( S \). Since \( \gamma \) can be analytically continued (a little) across the line segments \((-\pi - i \ln 2 \prec -\pi + i \ln 2\) and \((\pi - i \ln 2 \prec \pi + i \ln 2)\), and since \(-\pi\) and \(\pi\) are fixed points of the continued \( \gamma \), we see that power series expansions at 0 of the functions \( \gamma^n(z), \text{err}_n(z), \) and \( \tau_n(z) \) have convergence radii greater than \( \pi \).

The function \( \gamma \) has (in the strip \( S \)) a single zero 0; we now remove it. There is a unique function \( \varphi \in H(S) \) such that \( \gamma(z) = \frac{1}{2\pi} z^3 \varphi(z) \) for every \( z \in S \). The function \( \varphi \) has no zeros, it is even, and \( \varphi(0) = 1 \). Since \( S \) is a simply connected domain, \( \varphi(z) \) has a unique logarithm with the value 0 at \( z = 0 \), which means that there is a unique function \( \ln \varphi \in H(S) \) such that \( \ln \varphi(0) = 0 \) and \( \varphi(z) = \exp(\ln \varphi(z)) \) for every \( z \in S \).

We retreat, for a while, from the complex plane to the real axis. Given an angle \( \alpha \), \( 0 \leq \alpha \leq \pi \), we can visualise the sequence \( \alpha, \gamma(\alpha), \gamma(\gamma(\alpha)), \ldots, \) in the usual way:

\[
\begin{array}{c}
0 & \gamma(\alpha) & \pi 2 & \pi \\
\hline
\gamma(\gamma(\alpha)) & & & \\
0 & \gamma(\alpha) & \pi 2 & \pi \\
\end{array}
\]

The sequence in the figure starts at \( \alpha = 165^\circ \); we can follow it up to the second iteration, while all further iterations are hidden in the thick line representing the diagram of \( \gamma \). The sequence \( \alpha, \gamma(\alpha), \gamma^2(\alpha), \gamma^3(\alpha), \ldots \) is constant when \( \alpha = 0 \) or \( \alpha = \pi \), and is strictly decreasing if \( 0 < \alpha < \pi \) because \( 0 < \gamma(\xi) < \xi \) for every \( \xi \) in the range \( 0 < \xi < \pi \).

Let us express \( \gamma(\alpha), \gamma^2(\alpha), \gamma^3(\alpha), \ldots \) in terms of the function \( \varphi \):

\[
\gamma(\alpha) = \frac{1}{3} \alpha^3 \varphi(\alpha),
\gamma^2(\alpha) = \gamma(\gamma(\alpha)) = \frac{1}{3^4} \alpha^3 \varphi(\alpha)^3 \varphi(\gamma(\alpha)),
\gamma^3(\alpha) = \gamma(\gamma^2(\alpha)) = \frac{1}{3^6} \alpha^3 \varphi(\alpha)^3 \varphi(\gamma^2(\alpha))^3 \varphi(\gamma^3(\alpha)),
\ldots \ldots \ldots
\]

The general formula is

\[
\gamma^n(\alpha) = \frac{1}{3^{3n}(3^n-1)} \alpha^{3n} \prod_{k=0}^{n-1} \varphi(\gamma^k(\alpha))^{3^n-k-1}, \quad n \in \mathbb{N}.
\]

\[\text{For every } z \in S, \ln \varphi(z) = \int_0^z (\varphi'(w)/\varphi(w))dw, \text{ where the integration is along any curve in } S \text{ from } 0 \text{ to } z.\]
Glancing at the diagrams of \( \varphi \) (left panel) and its derivative \( \varphi' \) (right panel),

we see that \( \varphi(\xi) \) strictly increases for \( 0 \leq \xi \leq \pi \), and so we have the estimates

\[
\prod_{k=0}^{n-1} \varphi(\gamma^k(\alpha))^{3^{n-k-1}} \leq \prod_{k=0}^{n-1} \varphi(\alpha)^{3^{n-k-1}} = \varphi(\alpha)^{(3^n-1)/2},
\]

\[
\gamma^n(\alpha) < 3^{3/2} \varphi(\alpha)^{-1/2} \cdot \left( \alpha \cdot \left( \frac{\varphi(\alpha)}{3^{3/2}} \right)^{1/2} \right)^{2^n}
\]

This gives us the following upper bound for the error \( \text{err}_n(\alpha) \):

\[
\text{err}_n(\alpha) \leq D(\alpha) \cdot (\alpha d(\alpha))^{3^n}, \quad 0 \leq \alpha \leq \pi, \quad n \geq 1,
\]

where

\[
D(\alpha) = \sqrt{\frac{3}{\varphi(\alpha)}}, \quad d(\alpha) = \sqrt{\frac{\varphi(\alpha)}{27}}.
\]

When \( \alpha \) increases from 0 to \( \pi \), \( D(\alpha) \) decreases from \( \sqrt{3} \) to \( \pi/3 \) while \( d(\alpha) \) increases from \( 1/\sqrt{27} \) to \( 1/\pi \). At \( \alpha = 225^\circ = \pi/8 \) we estimate \( D(\pi/8) < 2 \) and \( (\pi/8) d(\pi/8) < 1/13 \) and thus obtain the upper bound

\[
\text{err}_n(\pi/8) < 2 \cdot 13^{-3^n}.
\]

How good (or bad) are the upper bounds we have obtained? Let us define

\[
q_n = D(\pi/8) \cdot ((\pi/8) d(\pi/8))^{3^n/\text{err}_n(\pi/8)}, \quad r_n = 2 \cdot 13^{-3^n/\text{err}_n(\pi/8)},
\]

and calculate \( \text{err}_n(\pi/8), q_n, \) and \( r_n \) for \( 1 \leq n \leq 8 \):

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{err}_n(\pi/8) )</th>
<th>( q_n )</th>
<th>( r_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 7.574 \cdot 10^{-4} )</td>
<td>1</td>
<td>1.202</td>
</tr>
<tr>
<td>2</td>
<td>( 1.448 \cdot 10^{-10} )</td>
<td>1.013</td>
<td>1.302</td>
</tr>
<tr>
<td>3</td>
<td>( 1.012 \cdot 10^{-30} )</td>
<td>1.053</td>
<td>1.657</td>
</tr>
<tr>
<td>4</td>
<td>( 3.457 \cdot 10^{-91} )</td>
<td>1.183</td>
<td>3.411</td>
</tr>
<tr>
<td>5</td>
<td>( 1.377 \cdot 10^{-272} )</td>
<td>1.677</td>
<td>2.977 \cdot 10^1</td>
</tr>
<tr>
<td>6</td>
<td>( 8.709 \cdot 10^{-817} )</td>
<td>4.776</td>
<td>1.979 \cdot 10^4</td>
</tr>
<tr>
<td>7</td>
<td>( 2.202 \cdot 10^{-2449} )</td>
<td>1.104 \cdot 10^2</td>
<td>5.810 \cdot 10^{12}</td>
</tr>
<tr>
<td>8</td>
<td>( 3.557 \cdot 10^{-7347} )</td>
<td>1.362 \cdot 10^6</td>
<td>1.471 \cdot 10^{38}</td>
</tr>
</tbody>
</table>

Can we do better than this? We can. We use (3) to obtain a tighter upper bound of \( \gamma^n(\alpha) \):

\[
\gamma^n(\alpha) = 3^{3/2} \left( \frac{\alpha}{3^{3/2}} \right)^{3^n} \left( \prod_{k=0}^{n-1} \varphi(\gamma^k(\alpha))^{3^{n-k-1}} \right)^{3^n} \leq 3^{3/2} \left( \frac{\alpha}{3^{3/2}} \right)^{3^n} \cdot (\psi(\alpha))^{3^n},
\]

where

\[
\psi(\alpha) = \prod_{k=0}^{\infty} \varphi(\gamma^k(\alpha))^{3^{k-1}}.
\]
The infinite product in (4) converges, and it does so uniformly, on the interval $[0 \ldots \pi]$, because $1 \leq \varphi(\gamma^k(\alpha))^{3^{-k-1}} \leq \varphi(\pi)^{3^{-k-1}}$ and $\prod_{k=0}^{\infty} \varphi(\pi)^{3^{-k-1}} = \varphi(\pi)^{1/2} = 3^{3/2}/\pi$. The product $\psi(\alpha)$ is therefore a continuous function on $[0 \ldots \pi]$, strictly increasing from $\psi(0) = 1$ to $\psi(\pi) = 3^{3/2}/\pi$:

![Graph of $\psi(\alpha)$ and $\sqrt{\varphi}$](image)

Now we have an upper bound of $err_n(\alpha)$,

$$err_n(\alpha) \leq \varepsilon_n(\alpha) := \sqrt[n]{3} (\alpha b(\alpha))^{3^n}, \quad b(\alpha) = \frac{\psi(\alpha)}{3\sqrt{3}}, \quad 0 \leq \alpha \leq \pi, \; n \geq 1,$$

which is tight indeed, because

$$\frac{\varepsilon_n(\alpha)}{err_n(\alpha)} = \psi(\gamma^n(\alpha)) \to 1 \text{ as } n \to \infty, \quad 0 < \alpha < \pi;$$

moreover, $\varepsilon_n(\alpha)/err_n(\alpha)$ converges to 1 very fast because

$$\psi(\gamma^n(\alpha)) = 1 + O(\gamma^n(\alpha)^2) = 1 + O\left((\alpha b(\alpha))^{2 \cdot 3^n}\right),$$

where $\alpha b(\alpha) < 1$ if $0 \leq \alpha < \pi$. Fix an $\alpha$, $0 < \alpha < \pi$. The upper bound $\varepsilon_n(\alpha)$ of $err_n(\alpha)$ is the best upper bound of $err_n(\alpha)$ of the form $C \cdot (\alpha c)^{3^n}$ with $C > 0$ and $c > 0$, meaning that if $c < b(\alpha)$, or $c = b(\alpha)$ and $C < \sqrt{3}$, then $C \cdot (\alpha c)^{3^n}$ is not an upper bound of $err_n(\alpha)$ for all $n$. In particular, $b(\alpha) \leq d(\alpha)$, that is, $\psi(\alpha) \leq \sqrt{\varphi(\alpha)},$

which we could derive directly, noting that $\varphi(\gamma^k(\alpha))^{3^{-k-1}} \leq \varphi(\pi)^{3^{-k-1}}$ for every $k \in \mathbb{N}$.

We have introduced the function $\psi(\alpha)$ only for positive $\alpha$ up to $\pi$, but it is in fact defined for every $\alpha$ in the interval $(-\pi \ldots \pi)$; it is an even function on this interval. Is $\psi(\alpha)$ an analytic function? We have defined it by a uniformly convergent infinite product of analytic functions, which, however, does not make it analytic. To exhibit a counterexample, think Fourier analysis: the series $\sum_{k=1}^{\infty} k^{-2} \sin(2^k t)$ uniformly converges on $\mathbb{R}$ to a continuous nowhere differentiable function; the infinite product $\prod_{k=1}^{\infty} \exp\left(k^{-2} \sin(2^k t)\right)$ therefore uniformly converges to a nowhere differentiable function.

The main purpose of our foray into the realm of complex numbers was to prepare grounds for the proof that the function $\psi(\alpha)$, as it is defined on the interval $(-\pi \ldots \pi)$, extends to a holomorphic function $\psi(z)$ defined on the strip $S$. We define $\psi(z)$ as

$$\psi(z) := \prod_{k=0}^{\infty} \varphi(\gamma^k(z))^{3^{-k-1}} = \prod_{k=0}^{\infty} \exp\left(3^{-k-1} \ln \varphi(\gamma^k(z))\right), \quad z \in S. \quad (5)$$
We claim that the infinite product in (5) converges uniformly on compact subsets of \( S \); when we prove this, it will follow that \( \psi \in H(S) \). Let \( K \) be a compact subset of \( S \). The set \( K \) is contained in some rectangle \( R(a, A(a)) \); since uniform conver- 

cence of the product in (5) on the rectangle implies its uniform conver- 

cence on \( K \), we can assume that \( K \) is the rectangle, and there- 

fore that \( \gamma(K) \subseteq K \). Since the function \( |\ln \varphi| \) is continuous on \( S \), its restriction to \( K \) attains a maximum value \( M \) at some point of \( K \). If \( z \in K \), then \( \gamma^k(z) \in K \), and therefore \( |\ln \varphi(\gamma^k(z))| \leq M \), for every \( k \in \mathbb{N} \), which shows that the infinite sum \( \sum_{k=0}^{\infty} 3^{-k-1} \ln \varphi(\gamma^k(z)) \) uniformly converges on \( K \), and the same is then true for the infinite product in (5). Done.

We can determine the coefficients of the power series expansion of \( \psi(z) \) at 0,

\[
\psi(z) = 1 + \sum_{k=1}^{\infty} c_k z^{2k},
\]

as many as we wish, from the functional equation

\[
\psi(z)^3 = \varphi(z) \psi(\gamma(z))
\]

that \( \psi(z) \) satisfies. Equating the power series coefficients on the two sides of (7) gives us recurrence relations

\[
c_k = f_k(c_1, \ldots, c_{k-1}), \quad k \geq 1,
\]

where \( f_1() = \frac{1}{36} \) and \( f_2(c_1) = \frac{7}{3240} - c_1^2 \), while for \( k > 2 \), \( f_k(c_1, \ldots, c_{k-1}) \) is a cubic polynomial in \( c_1, \ldots, c_{k-1} \) with rational coefficients. The power series expansion of \( \psi(z) \) begins

\[
\psi(z) = 1 + \frac{1}{36}z^2 + \frac{1}{720}z^4 + \frac{109}{1377810}z^6 + \frac{1811}{396809280}z^8 + \frac{264629}{1047576499200}z^{10} + \cdots.
\]

Not all coefficients are positive; of the first hundred coefficients \( c_k \), the coefficients with indices \( k \) in the ranges \( 22 \leq k \leq 26 \), \( 33 \leq k \leq 40 \), and \( 79 \leq k \leq 100 \) are negative. The convergence radius of the power series (6) is at least \( \pi \). In fact it is exactly \( \pi \): a little thought (with an ingredient of mathematical intuition) tells us that \( \psi(z) \) (or, to be precise, its analytic continuation) has rather nasty singularities at \( \pm \pi \pm i s_m \), \( m \in \mathbb{N} \), where \( s_0 = \ln 2 > s_1 > s_2 > \cdots > 0 \) and \( s_n \to 0 \) as \( n \to \infty \), thus \( \psi(z) \) has extremely nasty singularities at \( -\pi \) and \( \pi \). We may not have a formula for the general coefficient \( c_k \), but one thing we do know about the coefficients is that \( \limsup_{k \to \infty} 2^k |c_k| = 1/\pi \).

Let us use the tight upper bound \( \varepsilon_n(\alpha) \) of \( err_n(\alpha) \) to obtain better upper bounds of \( err_n(\pi/8) \) than we obtained before. Denoting by \( \psi_m(\alpha) \) the product of the first \( m \) factors (those with \( 0 \leq k \leq m - 1 \)) of the infinite product in (4), we find by direct calculation that \( \psi_1(\pi/8), \psi_2(\pi/8), \psi_3(\pi/8), \) and \( \psi_4(\pi/8) \) are precise, respectively, to 7, 21, 62, and 183 significant decimal places. Using \( \psi_2(\pi/8) \), we compute \((\pi/8)b(\pi/8)\) to 21 significant decimal places; rounding the eighth significant digit upwards, we get the upper bound

\[
err_n(\pi/8) \leq \sqrt{3} \cdot 0.075901232^{3n}, \quad n \geq 1.
\]

Setting \( p_n := \sqrt{3} \cdot 0.075901232^{3n}/err_n(\pi/8) \), we have

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000000181</td>
</tr>
<tr>
<td>2</td>
<td>1.000000112</td>
</tr>
<tr>
<td>3</td>
<td>1.000000336</td>
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<tr>
<td>4</td>
<td>1.000001009</td>
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<td>5</td>
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</tr>
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</tr>
<tr>
<td>7</td>
<td>1.000027237</td>
</tr>
<tr>
<td>8</td>
<td>1.000081713</td>
</tr>
</tbody>
</table>
and we see that the new upper bounds are indeed an improvement on the ones we produced before. However, further along we get $p_{17} = 4.994$, $p_{18} = 124.6$, and $p_{19} = 1.933 \cdot 10^8$, and after that things soon become very much worse. But this is in the nature of the fixed precision of the upper bound $0.075 \, 901 \, 232 > (\pi/8) b(\pi/8)$: if we have $c = \alpha b(\alpha) \cdot e^n$, with $n > 0$ no matter how small, then $\sqrt{3} \cdot c^{3n} \geq \sqrt{3} \cdot (\alpha b(\alpha))^{3n} \cdot e^{\eta^{3n}} \geq \text{err} n(\alpha) \cdot e^{\eta^{3n}}$, where $\eta^{3n}$ will sooner or later become arbitrarily large, growing very fast, and $e^{\eta^{3n}}$ will be growing faster still.

Suddenly, we are struck with a horrible suspicion: is it possible that our function $\psi(\alpha)$ is only a sleek theoretical chimera, which, however, is completely useless for actual numerical computations? Fortunately, it is not so. To illustrate numerical utility of the upper bounds, we urgently need to know $\text{err} 100(\pi/8)$ to ten significant decimal places (perhaps because, when written in scientific notation, it will give us the one-off decryption key for some extremely important encrypted communication).

We fire up MATHEMATICA to do it.

First we try to calculate $\text{err} 100(\pi/8) = \frac{1}{3} \gamma^{100}(\pi/8)$, computing values of the function $\gamma$ using its definition $\gamma(\alpha) = \alpha - 3 \arctan((\sin \alpha)/(2 + \cos \alpha))$, and notice that after a couple of iterations we start loosing precision at an alarming rate. The reason is that for very small $\alpha$ we compute $\gamma(\alpha)$ as the difference of two quantities that are very close together, relative to their size, and the smaller $\alpha$ is, the closer they are. To compute, say, $\text{err} 8(\pi/8) = 3.557 \cdot 10^{-7347}$ to ten significant decimal places, we have to begin the calculation with the precision of 7359 significant decimal places. To remedy this absurd loss of precision, we massage the formula for $\gamma(\alpha)$ into a numerically more stable form. We introduce shorthands $c = \cos(\alpha/3)$ and $s = \sin(\alpha/3)$, rewrite $\alpha$ as $3 \arctan(s/c)$, express $\sin \alpha$ and $\cos \alpha$ by $c$ and $s$ as $\sin \alpha = (4c^2 - 1) s$ and $\cos \alpha = 4c^3 - 3c$, convert the difference of arctans into a single arctan using the identity $\arctan x - \arctan y = \arctan((x - y)/(1 + xy))$, then simplify the fraction in the argument of the resulting arctan while replacing any $s^2$ we encounter with $1 - c^2$, to obtain, finally,

$$
\gamma(\alpha) = 3 \arctan \left( \frac{2s(1 - c)}{2c^2 + 2c - 1} \right) = 3 \arctan \left( \frac{4(\sin \frac{\alpha}{3} \cos \frac{\alpha}{3})^2}{\cos \frac{\alpha}{3} + 2 \cos \frac{\alpha}{3}} \right).
$$

This new formula for $\gamma(\alpha)$ is much better suited for numerical calculations:

![Graph showing precision loss as a function of $\alpha$](image)

MATHEMATICA loses less than half a decimal place of precision for any positive $\alpha$ up to $\pi/4$. We start computing the errors $\text{err}_n(\pi/8)$ afresh, with the initial precision of 65 significant decimal places (to be on the safe side). All goes well up to the seventeenth step, where we still manage to obtain $\text{err}_{17}(\pi/8) = 2.061 \, 584 \ldots \cdot 10^{-144 \, 604 \, 850}$, but when we try to compute $\text{err}_{18}(\pi/8)$, MATHEMATICA reports underflow. MATHEMATICA has an upper limit for the size of the exponent even for arbitrary-precision real numbers; in particular, no arbitrary-precision real number can be smaller than $6.423 \cdot 10^{-323 \, 228 \, 430}$. There is no way to adapt the formula for $\gamma(\alpha)$ to circumvent this obstacle. We must try something different.

And here is where the function $\psi(\alpha)$ and the tight upper bounds $\varepsilon_n(\alpha)$ come to the rescue. The upper bound $\varepsilon_{100}(\pi/8)$ is an extremely precise approximation of $\text{err}_{100}(\pi/8)$. Recall that $\varepsilon_n(\alpha) = \text{err}_n(\alpha) \cdot \psi(\gamma^n(\alpha))$, where $\psi(\alpha) = 1 + O(\alpha^2)$. To remove the uncertainty about the size of the constant multiplier hidden in the big oh term, we made this multiplier explicit: looking at the diagram of the quotient $(\psi(\alpha) - 1)/\alpha^2$ for $0 < \alpha \leq \pi$,
we see that it attains the maximum value at \( \alpha = \pi \); since \( (\psi(\pi) - 1)/\pi^2 < 1/15 \), we have \( 1 \leq \psi(\alpha) \leq 1 + \alpha^2/15 \) for \( 0 \leq \alpha \leq \pi \). If we compute \( \varepsilon_{100}(\pi/8) \) instead of \( \text{err}_{100}(\pi/8) \), then the relative error we make is less than \( (\gamma_{100}(\pi/8))^2/15 \), which is certainly less than \( (\gamma_{2}(\pi/8))^2/15 \approx 1.258 \cdot 10^{-20} \). Since we need to know \( \text{err}_{100}(\pi/8) \) only to ten significant decimal places, we will surely get that precision (and much larger precision, too, if we wished so) by calculating \( \varepsilon_{100}(\pi/8) \) instead. We shall not ask MATHEMATICA to evaluate the formula \( \sqrt{3}((\pi/8)b(\pi/8))^{100} \) for \( \varepsilon_{100}(\pi/8) \), because this would cause underflow. We avoid the underflow problem by evaluating the base 10 logarithm of \( \varepsilon_{100}(\pi/8) \):

\[
\log_{10} \varepsilon_{100}(\pi/8) = \frac{1}{2} \log_{10} 3 + 3^{100} \cdot \left( \log_{10}(\pi/8) + \log_{10} \psi(\pi/8) - \frac{3}{2} \log_{10} 3 \right).
\]

To estimate the necessary working precision, we first calculate \( \log_{10} \varepsilon_{100}(\pi/8) \) as a double precision floating-point value (using \( \psi_2 \) in place of \( \psi \)), and get \(-5.77 \cdot 10^{47}\). We therefore need 48 decimal places for the integer part of the logarithm, and 10, or better 11, decimal places for the mantissa; to be on the safe side, we calculate the logarithm to 65 significant decimal places (using \( \psi_4 \) in place of \( \psi \)). And here we proudly present the desired error, in its full glory:

\[
\text{err}_{100}(\pi/8) \approx 7.690351655 \cdot 10^{-577} 09458763832397881269698762985632406213034261.
\]